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Sequential maximum likelihood estimation for reflected generalized Ornstein–Uhlenbeck processes☆

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ABSTRACT

In this paper, we investigate a sequential maximum likelihood estimator of the unknown drift parameter for a class of reflected generalized Ornstein–Uhlenbeck processes driven by spectrally positive Lévy processes. In both of the cases of negative drift and positive drift, we prove that the sequential maximum likelihood estimator of the drift parameter is closed, unbiased, normally distributed and strongly consistent. Finally a numerical test is presented to illustrate the efficiency of the estimator.

1. Introduction

In this paper, we explore a sequential maximum likelihood estimator (SMLE) of the unknown drift parameter for the following reflected generalized Ornstein–Uhlenbeck process (RGOU) driven by a spectrally positive Lévy process:

$$\begin{aligned} &X_t = x_0 - \alpha \int_0^t X_s \mathrm{d}s + \sigma W_t + Y_t + L_t \geq b, \quad t > 0, \\ &x_0 \geq b, \end{aligned} \tag{1}$$

where $W = (W_t;\ t \geq 0)$ is a standard Brownian motion and $Y = (Y_t;\ t \geq 0)$ denotes a spectrally positive pure jump Lévy process (which is independent of W) on a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t;\ t \geq 0), \mathbb{P})$, where the filtration $\mathbb{F} = (\mathcal{F}_t;\ t \geq 0)$ satisfies the usual conditions.

We here call the constant $\alpha \in \mathbb{R}$ the drift parameter of the RGOU process $X = (X_t;\ t \geq 0)$, which is unknown in this paper. The positive constant σ is the given volatility rate. The nonnegative process $L = (L_t;\ t \geq 0)$ is called the regulator at the lower reflecting barrier $b \geq 0$ (see Harrison, 1986). By virtue of Ata et al. (2005), the paths of the regulator L are nondecreasing, r.c.l.l. (right continuous with left limits) and possess the following support property:

$$L_t = \int_0^t \mathbb{1}_{\{X_s = b\}} \mathrm{d}L_s \tag{2}$$

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for all positive t, where $\mathbb{1}_A$ denotes the indicator associated to the set A. Due to the appearance of the regulator L in (1), the RGOU process $X_t \geq b$ for all positive t. In general, the regulator L has jumps, whose sizes can be identified by

$$\Delta L_t := L_t - L_{t-} = (b - \Delta Y_t - X_{t-})^+, \tag{3}$$

for all positive t. However, in our setting of the Lévy process Y, the jump's amplitude of Y, $\Delta Y_t := Y_t - Y_{t-} \geq 0$ for all positive t. Hence ΔL_t vanishes because $X_{t-} \geq b$ for all positive t. This shows that there exists a continuous modification to the regulator L. We here use this continuous modification in (1) and yet denote it by L.

The aim of this paper is to study the statistical inference for the RGOU process X given by (1). More precisely, we would estimate the unknown drift α in (1) based on a continuous observation of the state process X up to a certain predetermined level of precision by proposing to use a SMLE. Similarly to the case of MLE, the SMLE is carried out in the second-order sense. Hence we here assume that $\mathbb{E}[Y_1^2] < \infty$ throughout the paper. The MLE for the drift parameter of ROU processes driven by Brownian motions was first studied by Bo et al. (2011b). However, as pointed out by Lee et al. (2012), the MLE is not unbiased (see also Theorem 3.3 in Bo et al., 2011b). They proposed a SMLE of the unknown drift of the ROU process without jumps. Accordingly, a natural idea is to extend the SMLE of the drift of the continuous ROU process to that of the ROU process with jumps. In our jump's case, we prove that the SMLE associated with the unknown drift is closed, unbiased, normally distributed and strongly consistent. On the other hand, the reflected jump-diffusion (or Lévy) processes have been extensively investigated in the literature (Asmussen et al., 2004; Asmussen and Pihlsgard, 2007; Atar and Budhiraja, 2002; Avram et al., 2004, 2007; Bo et al., 2011a, in press, 2012; Xing et al., 2009), where the stationary property of the RGOU process (1) was discussed in Xing et al. (2009) when the drift $\alpha > 0$. As far as we know, there is no relevant research concerning the SMLE for the RGOU process (1).

The rest of the paper is organized as follows. We prove a relationship between the regulator and the local time of the RGOU process in Section 2. We then establish the SMLE of the unknown drift of the RGOU process in terms of the local time and the observed process itself in Section 3. In the same section, we prove that the SMLE of the drift is closed, unbiased, normally distributed and strongly consistent. At the end of this section, the case of two-sided reflection is also presented as a remark. Finally, we present some numerical illustrations for the efficiency of the proposed estimator in the last section.

Throughout the paper, we use the conventions:

$$\int_c^d = \int_{(c,d]}, \quad \text{and} \quad \int_c^\infty = \int_{(c,\infty)},$$

for any real numbers $c \leq d$.

2. Regulator and local time of RGOU process

It is well known that the pure jump Lévy process $Y = (Y_t;\ t \geq 0)$ admits the following Lévy–Itô decomposition:

$$Y_t = Y_0 + mt + \int_0^t \int_{\{y>1\}} yN(\mathrm{d}y, \mathrm{d}s) + \int_0^t \int_{\{0<y\leq 1\}} y\widetilde{N}(\mathrm{d}y, \mathrm{d}s),$$

where $Y_0 \in \mathbb{R}$ is the initial value of Y, $m \in \mathbb{R}$, the Poisson random measure N has the representation:

$$N(A \times (0, t]) = \sum_{s\leq t} \mathbb{1}_{\{\Delta Y_s \in A\}}, \quad \forall A \in \mathcal{B}(\mathbb{R}_+),$$

and the corresponding compensated Poisson random measure $\widetilde{N}$ with compensator ν on $\mathcal{B}(\mathbb{R}_+)$ is given by

$$\widetilde{N}(A \times (0, t]) = N(A \times (0, t]) - t\nu(A).$$

The following result shows that the regulator L in (1) is closely related to the local time of the RGOU process X at point b.

Proposition 1. *Let $\ell^b = (\ell_t^b;\ t \geq 0)$ represent the local time process for the semimartingale X given by (1) at point b. Then the regulator L has the following expression in term of the local time ℓ^b:*

$$\begin{aligned} L_t &= \frac{1}{2}\ell_t^b - \alpha b \int_0^t \mathbb{1}_{\{X_s=b\}} \mathrm{d}s \\ &= \frac{1}{2}\ell_t^b, \quad \forall\, t \geq 0, \end{aligned} \tag{4}$$

where $\alpha \in \mathbb{R}$ is the drift parameter of the RGOU process X given by (1) and $b \geq 0$ is the lower reflecting barrier of X.

Proof. Apply Tanaka's formula to the semimartingale X, we conclude that

$$(X_t-b)^+=(x_0-b)^+ + \int_0^t \mathbb{1}_{\{X_{s-}>b\}}\mathrm{d}X_s + \sum_{0<s\le t}\mathbb{1}_{\{X_{s-}>b\}}(X_s-b)^- + \sum_{0<s\le t}\mathbb{1}_{\{X_{s-}\le b\}}(X_s-b)^+ + \frac{1}{2}\ell_t^b. \tag{5}$$

Noting that $X_t \ge b$ for all $t \ge 0$. Then the 2nd line of the above display vanishes and the r.h.s. of the 1st line of the above display equals

$$X_t - x_0 - \int_0^t \mathbb{1}_{\{X_{s-}=b\}}\mathrm{d}X_s.$$

The 4th term of r.h.s. of (5) reduces to

$$\sum_{0<s\le t}\mathbb{1}_{\{X_{s-}=b\}}(X_s-b).$$

Let $X_t^c = X_t - \sum_{s\le t}\Delta X_s$ be the continuous part of the semimartingale X. Then

$$X_t - b = x_0 - b + X_t - x_0 - \int_0^t \mathbb{1}_{\{X_{s-}=b\}}\mathrm{d}X_s^c - \sum_{0<s\le t}\mathbb{1}_{\{X_{s-}=b\}}\Delta X_s + \sum_{0<s\le t}\mathbb{1}_{\{X_{s-}=b\}}(X_s-b) + \frac{1}{2}\ell_t^b,$$

which leads to the following equality by a simple calculation:

$$\int_0^t \mathbb{1}_{\{X_{s-}=b\}}\mathrm{d}X_s^c = \frac{1}{2}\ell_t^b.$$

By employing (2), it follows that

$$\sigma\int_0^t \mathbb{1}_{\{X_s=b\}}\mathrm{d}W_s = \frac{1}{2}\ell_t^b - L_t - \alpha b\int_0^t \mathbb{1}_{\{X_s=b\}}\mathrm{d}s,$$

which further implies that $\sigma\int_0^t \mathbb{1}_{\{X_s=b\}}\mathrm{d}W_s = 0$ for all $t \ge 0$, since a continuous local martingale with finite variation equals its initial value. Finally we verify that the time of the RGOU process X spent on the barrier b has Lebesgue measure zero. In fact, as a simple consequence of the occupation time formula (see Page 219 in Protter, 2005), we have $\mathbb{P}$-a.s.

$$\int_0^t \mathbb{1}_{\{X_s=b\}}\mathrm{d}[X^c,X^c]_s = \int_{\mathbb{R}} \mathbb{1}_{\{a=b\}}\ell_t^a\mathrm{d}a = \ell_t^b\int_{\mathbb{R}}\mathbb{1}_{\{a=b\}}\mathrm{d}a = 0, \quad \forall\, t\ge 0.$$

Note that $[X^c,X^c]_t = \sigma^2 t$ for all positive t. The above equality yields that

$$\int_0^t \mathbb{1}_{\{X_s=b\}}\mathrm{d}s = 0, \quad \mathbb{P}\text{-a.s.},$$

for all $t \ge 0$. This proves the equality (4). □

Remark 2. Compared with the regulator L satisfying the property (2), the local time ℓ^b (and hence L) can be identified by the paths of the semimartingale X as follows:

$$\begin{aligned}\ell_t^b &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_0^t \mathbb{1}_{\{b\le X_s\le b+\varepsilon\}}\mathrm{d}[X^c,X^c]_s\\ &= \lim_{\varepsilon\to 0}\frac{\sigma^2}{\varepsilon}\int_0^t \mathbb{1}_{\{X_s\le b+\varepsilon\}}\mathrm{d}s, \quad \mathbb{P}\text{-a.s.}, \ \forall\, t\ge 0,\end{aligned} \tag{6}$$

since $X_t \ge b$ and $[X^c,X^c]_t = \sigma^2 t$ for all positive t (see Page 230 in Protter, 2005).

3. Sequential maximum likelihood estimation

In this section, we consider using the SMLE $\left(\tau(H), \widehat{\alpha}_{\tau(H)}\right)$ (see Theorem 3) to estimate the unknown drift α of the RGOU process X described as (1). Here, the random time $\tau(H)$ is defined to be a stopping time which is the first time such that the observed Fisher information of the RGOU process exceeds a predetermined level of precision H and $\widehat{\alpha}_{\tau(H)}$ is a sequential estimator of the drift α tracking at $\tau(H)$.

More specifically, the tracking stopping time is defined to be

$$\tau(H) = \inf\left\{t>0;\ \int_0^t |X_s|^2\mathrm{d}s \ge H\right\}, \tag{7}$$

where the predetermined level of precision $0 < H < \infty$, which is $\mathcal{F}_{\tau(H)}$-measurable (see, e.g., Problem 2.13 on Page 8 in Karatzas and Shreve, 1991). The sequential estimator $\left(\tau(H), \widehat{\alpha}_{\tau(H)}\right)$ will be established in the following main theorem of this paper.

Theorem 3. *Let the random pair $(\tau(H), \widehat{\alpha}_{\tau(H)})$ be the sequential estimation plan with the observation stopping time $\tau(H)$ defined as (7). Then the SMLE of the unknown drift α is established by*

$$\begin{aligned}\widehat{\alpha}_{\tau(H)} &= \frac{1}{H}\left[bL_{\tau(H)} - \int_0^{\tau(H)} X_t \mathrm{d}X_t^c\right], \\ &= \frac{1}{H}\left[\frac{b}{2}\ell^b_{\tau(H)} - \int_0^{\tau(H)} X_t \mathrm{d}X_t^c\right],\end{aligned} \tag{8}$$

where L is the regulator of the RGOU process X and X^c is the continuous counterpart of X. The process ℓ^b is the local time of X at point b (see Proposition 1 and Remark 2 in Section 1). Moreover, we have

(i) *The sequential estimator $\left(\tau(H), \widehat{\alpha}_{\tau(H)}\right)$ established as above is closed. In other words, for each $H \in (0, \infty)$,*

$$\mathbb{P}_\alpha\left(\tau(H) < \infty\right) = 1 \quad \text{for all } \alpha \in (-\infty, \infty),$$

where $\mathbb{P}_\alpha$ is the probability measure induced by the RGOU process X given by (1) with drift α.

(ii) *The sequential estimator is unbiased, i.e., for each $H \in (0, \infty)$,*

$$\mathbb{E}_\alpha\left[\widehat{\alpha}_{\tau(H)}\right] = \alpha, \quad \text{for all } \alpha \in (-\infty, \infty),$$

where $\mathbb{E}_\alpha$ denotes the expectation operator corresponding to the probability measure $\mathbb{P}_\alpha$.

(iii) *For each $H > 0$ fixed, it holds that*

$$\sqrt{H}\left[\widehat{\alpha}_{\tau(H)} - \alpha\right] \sim \mathrm{N}(0, \sigma^2),$$

where $\mathrm{N}(0, \sigma^2)$ denotes the normal distribution with mean zero and variance $\sigma^2 > 0$.

(iv) *The sequential estimator is strongly consistent. Namely*

$$\widehat{\alpha}_{\tau(H)} \to \alpha, \quad \mathbb{P}\text{-a.s.}$$

when $H \to \infty$.

Remark 4. If the lower reflecting barrier $b = 0$ (see, e.g., Asmussen et al., 2004; Asmussen and Pihlsgard, 2007; Avram et al., 2004, 2007; Xing et al., 2009), i.e. the RGOU process X reflects at barrier 0, then in this case, the SMLE (8) is given by

$$\widehat{\alpha}_{\tau(H)} = -\frac{1}{H}\int_0^{\tau(H)} X_t \mathrm{d}X_t^c,$$

where $0 < H < +\infty$.

Proof of Theorem 3. Let θ, α be any two real numbers. Suppose that (X^θ, L^θ) and (X^α, L^α) satisfy the following reflected stochastic differential equations (RSDEs), respectively:

$$\begin{aligned}&\mathrm{d}X_t^k = -kX_t^k \mathrm{d}t + \sigma \mathrm{d}W_t + \mathrm{d}Y_t + \mathrm{d}L_t^k, \\ &L_t^k = \int_0^t \mathbb{1}_{\{X_s^k = b\}} \mathrm{d}L_s^k, \\ &X_t^k \geq b, \quad \forall\, t \geq 0,\end{aligned}$$

where $k \in \{\theta, \alpha\}$. For each $k \in \{\theta, \alpha\}$, let $\mathbb{P}^k_{\tau(H),X}$ denote the induced probability measure by the RGOU process $X^{k,\tau(H)} = (X^k_{t\wedge\tau(H)};\ t \geq 0)$ on the Skorohod space $D(\mathbb{R}_+; \mathbb{R}_+)$. Then $\mathbb{P}^\theta_{\tau(H),X}$ and $\mathbb{P}^\alpha_{\tau(H),X}$ are mutually equivalent and the corresponding Radon–Nikodym derivative is given by

$$\left.\frac{\mathbb{P}^\alpha_{\tau(H),X}}{\mathbb{P}^\theta_{\tau(H),X}}\right|\mathcal{F}^{X^\theta}_{\tau(H)} = \exp\left[-\frac{\alpha - \theta}{\sigma}\int_0^{\tau(H)} X_t^\theta \mathrm{d}W_t - \frac{(\alpha - \theta)^2}{2\sigma^2}\int_0^{\tau(H)} \left|X_t^\theta\right|^2 \mathrm{d}t\right]. \tag{9}$$

Note that $\mathbb{P}^0_{\tau(H),X}$ corresponds to the induced probability measure by the reflected Lévy process $X^0 = (X_t^0 := x_0 + \sigma W_t + Y_t + L_t;\ t \geq 0)$ on the Skorohod space. Then the Log likelihood function $l(\alpha)$ associated to the unknown α is

$$l(\alpha) = \sigma^2 \log\left[\left.\frac{\mathbb{P}^\alpha_{\tau(H),X}}{\mathbb{P}^0_{\tau(H),X}}\right|\mathcal{F}^X_{\tau(H)}\right]$$

$$= -\alpha \int_0^{\tau(H)} X_t \mathrm{d}X_t^c + \alpha \int_0^{\tau(H)} X_t \mathrm{d}L_t - \frac{\alpha^2}{2} \int_0^{\tau(H)} X_t^2 \mathrm{d}t$$
$$= -\alpha \int_0^{\tau(H)} X_t \mathrm{d}X_t^c + \alpha b L_{\tau(H)} - \frac{\alpha^2}{2} H, \tag{10}$$

where we have used $X = (X_t;\ t \geq 0)$ to denote the RGOU process $X^\alpha = (X_t^\alpha;\ t \geq 0)$ described as (1) with the drift α for convenience. Hence the sequential estimator (8) is derived by solving the equation $\frac{\mathrm{d}}{\mathrm{d}\alpha} l(\alpha) = 0$ w.r.t. the unknown variable α.

To verify (i), it suffices to prove $\mathbb{P}^\alpha \left(\int_0^\infty |X_t|^2 \mathrm{d}t = \infty \right) = 1$. We first consider the case of $\alpha > 0$. Since $\mathbb{E}_\alpha[|Y_1|^2] < \infty$, which is equivalent to the condition $\int_1^\infty y^2 \nu(\mathrm{d}y) < \infty$, we hence have

$$\int_1^\infty y \nu(\mathrm{d}y) \leq \int_1^\infty y^2 \nu(\mathrm{d}y) < \infty.$$

Thus the ergodic conditions of the RGOU process in Theorem 2.1 of Xing et al. (2009) are satisfied, and hence there exists a unique stationary distribution ϕ_∞ on $\mathcal{B}(\mathbb{R}_+)$ for the RGOU process (1). By the ergodic theorem, we get

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T |X_t|^2 \mathrm{d}t = \int_{[b,\infty)} x^2 \phi_\infty(\mathrm{d}x) > 0,$$

which implies that $\mathbb{P}^\alpha \left(\int_0^\infty |X_t|^2 \mathrm{d}t = \infty \right) = 1$. As for the case of $\alpha < 0$, the RGOU process is not ergodic. However we can use the following comparison property of RGOU processes with different drifts to deal with this case. Namely, suppose that (X^1, L^1) and (X^2, L^2) solve the following RSDEs respectively:

$$\begin{aligned} &\mathrm{d}X_t^j = a_j X_t^j \mathrm{d}t + \sigma \mathrm{d}W_t + \mathrm{d}Y_t + \mathrm{d}L_t^j, \\ &L_t^j = \int_0^t \mathbb{1}_{\{X_s^j = b\}} \mathrm{d}L_s^j, \\ &X_t^j \geq b, \quad \forall\, t \geq 0, \end{aligned} \tag{11}$$

where $j \in \{1, 2\}$, $b \leq X_0^1 \leq X_0^2$ and $-\infty < a_1 < a_2 < +\infty$.

Now let $\eta(t) = [(X_t^1 - X_t^2)^+]^2$. Then by Tanaka's formula and note that the difference $X_t^1 - X_t^2$ does not consist of Brownian component, we have

$$\eta(t) = 2 \int_0^t (X_s^1 - X_s^2)^+ (a_1 X_s^1 - a_2 X_s^2) \mathrm{d}s + 2 \int_0^t (X_s^1 - X_s^2)^+ \mathrm{d}(L_s^1 - L_s^2).$$

Using (11), the 2nd term of r.h.s. of the above display is equal to

$$\begin{aligned} 2 \int_0^t (X_s^1 - X_s^2)^+ \mathrm{d}L_s^1 - 2 \int_0^t (X_s^1 - X_s^2)^+ \mathrm{d}L_s^2 &= 2 \int_0^t (b - X_s^2)^+ \mathrm{d}L_s^1 - 2 \int_0^t (X_s^1 - b)^+ \mathrm{d}L_s^2 \\ &= -2 \int_0^t (X_s^1 - b)^+ \mathrm{d}L_s^2 \leq 0, \end{aligned}$$

by employing the fact that $(b - X_t^2)^+ = 0$ (since $X_t^2 \geq b$ for all $t \geq 0$). Moreover, we have

$$\begin{aligned} 2 \int_0^t (X_s^1 - X_s^2)^+ (a_1 X_s^1 - a_2 X_s^2) \mathrm{d}s &= 2 \int_0^t (X_s^1 - X_s^2)^+ (a_1 X_s^1 - a_1 X_s^2) \mathrm{d}s + 2 \int_0^t (X_s^1 - X_s^2)^+ (a_1 X_s^2 - a_2 X_s^2) \mathrm{d}s \\ &\leq 2a_1 \int_0^t (X_s^1 - X_s^2)^+ (X_s^1 - X_s^2) \mathrm{d}s, \end{aligned}$$

since $X_t^2 \geq b \geq 0$ for all $t \geq 0$. We finally conclude that

$$0 \leq \eta(t) \leq 2a_1 \int_0^t \eta(s) \mathrm{d}s, \quad \text{for all positive } t.$$

Then $\eta(t) = 0$, $\mathbb{P}_\alpha$-a.s. by using the Gronwall's inequality. This further yields that

$$0 \leq b \leq X_t^1 \leq X_t^2, \quad \mathbb{P}_\alpha\text{-a.s.},$$

for all $t \geq 0$. For the same initial date $X_0^1 = X_0^2 = x_0 \geq b$, this comparison property shows that the RGOU process (1) with positive drift is always not smaller than the one with negative drift. Hence it also holds that $\mathbb{P}^\alpha \left(\int_0^\infty |X_t|^2 \mathrm{d}t = \infty \right) = 1$ in the case of $\alpha < 0$.

By the form of the sequential estimator (8) and noting that

$$\mathrm{d}X_t^c = -\alpha X_t \mathrm{d}t + \sigma \mathrm{d}W_t + \mathrm{d}L_t,$$

we arrive at

$$\begin{aligned} \widehat{\alpha}_{\tau(H)} &= \frac{1}{H}\left[bL_{\tau(H)} - \int_0^{\tau(H)} X_t \mathrm{d}X_t^c\right] \\ &= \frac{1}{H}\left[bL_{\tau(H)} + \alpha\int_0^{\tau(H)} |X_t|^2 \mathrm{d}t - \sigma\int_0^{\tau(H)} X_t \mathrm{d}W_t - \int_0^{\tau(H)} X_t \mathrm{d}L_t\right] \\ &= \alpha - \frac{\sigma}{H}\int_0^{\tau(H)} X_t \mathrm{d}W_t, \end{aligned}$$

where we used the following equality by the property (2):

$$\int_0^{\tau(H)} X_t \mathrm{d}L_t = bL_{\tau(H)}.$$

By Lemma 2.9.2 in Prakasa Rao (1999), the process $(\int_0^{\tau(H)} X_t \mathrm{d}W_t;\ H \geq 0)$ is a $\mathbb{P}_\alpha$-standard Brownian motion. Then the sequential estimator (8) is unbiased, i.e., (ii) holds. On the other hand, the random variable $\int_0^{\tau(H)} X_t \mathrm{d}W_t \sim \mathrm{N}(0, H)$, since $\int_0^{\tau(H)} |X_s|^2 \mathrm{d}s = H$ which proves that $\sqrt{H}[\widehat{\alpha}_{\tau(H)} - \alpha] \sim \mathrm{N}(0, \sigma^2)$ for each $H > 0$ fixed, i.e., (iii) is valid. Finally the conclusion (iv) follows from $\frac{1}{H}\int_0^{\tau(H)} X_t \mathrm{d}W_t \to 0$ as $H \to \infty$ by the law of large number. □

Remark 5. For the case of two-reflecting barriers, the RGOU process can be described as

$$\begin{aligned} &X_t = x_0 - \alpha\int_0^t X_s \mathrm{d}s + \sigma W_t + Y_t + L_t - U_t \in [b, d], \quad t > 0, \\ &x_0 \in [b, d], \end{aligned} \tag{12}$$

where $U = (U_t;\ t \geq 0)$ denotes the reflecting upper barrier d with $d > b$. The regulator U is a nondecreasing, r.c.l.l. process with $U_0 = 0$ and it satisfies that

$$U_t = \int_0^t \mathbb{1}_{\{X_s = d\}} \mathrm{d}U_s, \quad \forall\, t > 0.$$

Differently from the regulator L at lower reflecting barrier, the regulator U indeed has jumps, whose sizes can be given by

$$\Delta U_t := U_t - U_{t-} = (\Delta Y_t + X_{t-} - d)^+, \quad t \geq 0,$$

where we set $U_0 = U_{0-} = 0$ by convention. Let $U_t^c = U_t - \sum_{s \leq t} \Delta U_s$ be the continuous component of U. Then it also holds that

$$U_t^c = \int_0^t \mathbb{1}_{\{X_s = d\}} \mathrm{d}U_s^c, \quad \forall\, t > 0.$$

Similarly to the likelihood function given by (10), the likelihood function in the case of two-sided reflection admits the form:

$$\begin{aligned} l(\alpha) &= -\alpha\int_0^{\tau(H)} X_t \mathrm{d}X_t^c + \alpha\int_0^{\tau(H)} X_t \mathrm{d}L_t - \alpha\int_0^{\tau(H)} X_t \mathrm{d}U_t^c - \frac{\alpha^2}{2}\int_0^{\tau(H)} X_t^2 \mathrm{d}t \\ &= -\alpha\int_0^{\tau(H)} X_t \mathrm{d}X_t^c + \alpha bL_{\tau(H)} - \alpha \mathrm{d}U_{\tau(H)}^c - \frac{\alpha^2}{2}H. \end{aligned}$$

Therefore, the SMLE of the unknown drift α is given by

$$\begin{aligned} \widehat{\alpha}_{\tau(H)} &= \frac{1}{H}\left[bL_{\tau(H)} - \mathrm{d}U_{\tau(H)}^c - \int_0^{\tau(H)} X_t \mathrm{d}X_t^c\right], \\ &= \frac{1}{H}\left[\frac{b}{2}\ell_{\tau(H)}^b - \frac{d}{2}\ell_{\tau(H)}^d - \int_0^{\tau(H)} X_t \mathrm{d}X_t^c\right], \end{aligned} \tag{13}$$

where $H > 0$ and $\ell^d = (\ell_t^d;\ t \geq 0)$ is the local time of the RGOU process at point d. The equality (13) follows from the relationship $U_t^c = \frac{1}{2}\ell_t^d$ for all $t \geq 0$, whose proof is very similar to that of Proposition 1.

Table 1
The ME of the SMLE $\widehat{\alpha}_{\tau(H)}$, the MSE of the SMLE $\widehat{\alpha}_{\tau(H)}$ and the SD of $|\widehat{\alpha}_{\tau(H)} - \alpha|^2$.

Parameters	H	$\mathbb{E}[\tau(H)]$	$\mathbb{E}[\widehat{\alpha}_{\tau(H)} - \alpha]$	$\mathbb{E}[\lvert\widehat{\alpha}_{\tau(H)} - \alpha\rvert^2]$	SD of $\lvert\widehat{\alpha}_{\tau(H)} - \alpha\rvert^2$
$\alpha = 0.5$	10	3.745	−0.0019	0.0987	0.1390
$\sigma = 1$	50	9.864	+0.0012	0.0199	0.0284
$\gamma = 0.25$	100	16.46	+0.0009	0.0101	0.0141
$\alpha = 1$	10	3.800	−0.0062	0.4030	0.5792
$\sigma = 2$	50	13.32	+0.0043	0.0796	0.1113
$\gamma = 0.25$	100	24.77	−0.0016	0.0392	0.0554
$\alpha = 0.5$	10	2.145	−0.0079	0.0973	0.1408
$\sigma = 1$	50	4.591	−0.0019	0.0202	0.0280
$\gamma = 0.5$	100	6.714	−0.0014	0.0101	0.0146
$\alpha = -0.5$	10	1.938	+0.0126	0.0952	0.1335
$\sigma = 1$	50	2.903	+0.0092	0.0193	0.0267
$\gamma = 0.25$	100	3.368	+0.0080	0.0097	0.0137

We also note that the RGOU process (12) has a compact state space $[b, d]$. The compactness of the state space is sufficient for the existence of a stationary distribution ϕ_∞ on $\mathcal{B}([b, d])$ of the Markov process (12) (see Pages 2390–240 in Ethier and Kurtz, 1986). By virtue of the ergodic theorem, we have

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T |X_t|^2 \mathrm{d}t = \int_{[b,d]} x^2 \phi_\infty(\mathrm{d}x) > 0,$$

which implies that $\mathbb{P}^\alpha\left(\int_0^\infty |X_t|^2 \mathrm{d}t = \infty\right) = 1$. This proves that the conclusion (i) in Theorem 3 is also valid in the case of two-sided reflection. Similarly to the proof of Theorem 3, the conclusions (ii)–(iv) therein also hold in the case of two-sided reflection.

4. Numerical illustrations

In this section, we present some numerical illustrations to exhibit the performance of the SMLE $(\tau(H), \widehat{\alpha}_{\tau(H)})$ given by Theorem 3. In most practical applications the reflecting lower barrier is usually taken as $b = 0$ (see, e.g., Asmussen et al., 2004; Asmussen and Pihlsgard, 2007; Avram et al., 2004, 2007; Xing et al., 2009). So we here set $b = 0$. From Remark 4, it follows that the SMLE (8) becomes

$$\widehat{\alpha}_{\tau(H)} = -\frac{1}{H}\int_0^{\tau(H)} X_t \mathrm{d}X_t^c, \quad 0 < H < +\infty.$$

For the pure jump Lévy process $Y = (Y_t;\ t \geq 0)$, we take in this section

$$Y_t = \sum_{i=0}^{N_t} J_i, \quad t > 0, \text{ and } Y_0 = 0,$$

where $N = (N_t;\ t \geq 0)$ is a Poisson process with intensity $\lambda > 0$ and $(J_i;\ i = 1, 2, \ldots)$ are i.i.d. random variables with exponential density $f_J(x) = \frac{1}{\gamma}e^{-\frac{x}{\gamma}}\mathbb{1}_{\{x>0\}}$ which is also independent of N (here $\gamma > 0$).

We first simulate the sample paths of the ROU process (1) by using the Monte Carlo method. Here the numerical scheme presented in Lépingle (1995) is adopted, which has the same rate of convergence as the classic Euler–Maruyama scheme. In each numerical experiment, we generate 10^4 sample paths with step size $\Delta t = 10^{-2}$. We examine the following four different settings, respectively:

C1: $\alpha = 0.5, \sigma = 1, \lambda = 5$ and $\gamma = 0.25$,
C2: $\alpha = 1, \sigma = 2, \lambda = 5$ and $\gamma = 0.25$,
C3: $\alpha = 0.5, \sigma = 1, \lambda = 5$ and $\gamma = 0.5$,
C4: $\alpha = -0.5, \sigma = 1, \lambda = 5$ and $\gamma = 0.25$.

Table 1 reports some statistics related to the SMLE $\widehat{\alpha}_{\tau(H)}$, which include the Mean Error (ME) $\mathbb{E}[\widehat{\alpha}_{\tau(H)} - \alpha]$, the Mean Square Error (MSE) $\mathbb{E}[|\widehat{\alpha}_{\tau(H)} - \alpha|^2]$ and the Standard Deviation (SD) of $|\widehat{\alpha}_{\tau(H)} - \alpha|^2$. The mean time $\mathbb{E}[\tau(H)]$ needed to achieve three different H-levels are also reported.

Figs. 1–3 are plotted under the setting C1. Fig. 1 displays the MSE (the dashed line) of the SMLE $\widehat{\alpha}_{\tau(H)}$ against $H > 0$. The solid line in this plot is the theoretical MSE curve of the SMLE, i.e., σ^2/H. We observe that the two lines almost coincide. Fig. 2 shows the bias of the SMLE (from Theorem 3(iii), we have the SMLE $\widehat{\alpha}_{\tau(H)} - \alpha \sim \mathrm{N}(0, \sigma^2/H)$). Fig. 3 depicts the histogram of the statistic $\sqrt{H}[\widehat{\alpha}_{\tau(H)} - \alpha]/\sigma$ with $H = 10$ and $H = 50$. The dashed curve is the standard normal density. We find that the SMLE works quite well (from Theorem 3(iii), we have $\sqrt{H}[\widehat{\alpha}_{\tau(H)} - \alpha]/\sigma \sim \mathrm{N}(0, 1)$).

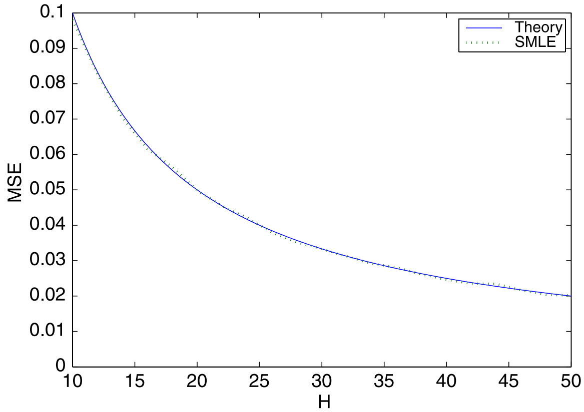

Fig. 1. The dashed line is the MSE plot for the SMLE $\widehat{\alpha}_{\tau(H)}$, and the solid line is the corresponding theoretical MSE of the SMLE.

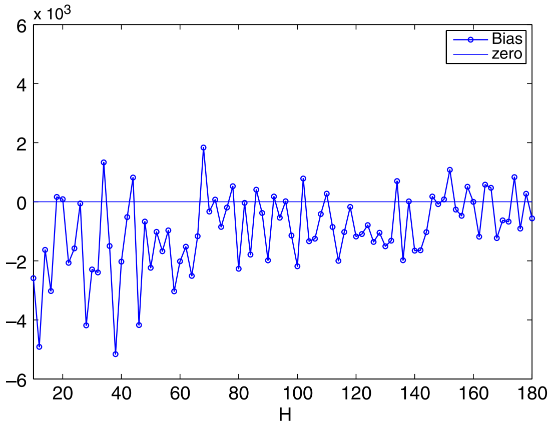

Fig. 2. Bias.

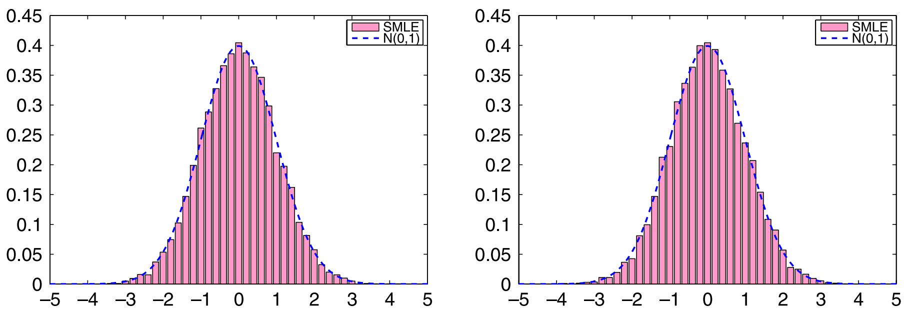

Fig. 3. Histogram of $\sqrt{H}[\widehat{\alpha}_{\tau(H)} - \alpha]/\sigma$ with $H = 10$ (left) and $H = 50$ (right). The dashed lines are the plots of the standard normal density.

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